The non-trivial zeros of Riemann's zeta-function

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Abstract

A proof of the Riemann hypothesis using the reflection principle $\zeta(\overline{s}) = \overline{\zeta(s)}$ is presented.

1 Introduction

The Riemann zeta-function $\zeta(s)$ can be defined by either of two following formulae [1]

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

where $n \in \mathbb{N}$, and

$$\zeta(s) = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} \tag{2}$$

where p runs through all primes and $\Re(s) > 1$. By analytic continuation $\zeta(s)$ is defined over whole $\mathbb{C}$.

The relationship between $\zeta(s)$ and $\zeta(1-s)$

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{3}$$

is known as the functional equation of the zeta-function. From the functional equation it follows that $\zeta(s)$ has zeros at $s = -2, -4, -6, \ldots$. These zeros are traditionally called trivial zeros of $\zeta(s)$; the zeros of $\zeta(s)$ with $\Im(z) \neq 0$ are called non-trivial zeros. From the equation (2), which is known as Euler's product, it was deduced that $\zeta(s)$ has no zeros for $\Re(s) > 1$. The functional equation implies that there are no non-trivial zeros with $\Re(s) < 0$. It was deduced that there are no zeros for $\Re(s) = 0$ and $\Re(s) = 1$. Therefore all non-trivial zeros are in the *critical strip* specified by $0 < \Re(s) < 1$.

Since that $\zeta(s)$ is real on the real axis we have by the reflection principle

$$\zeta(\overline{s}) = \overline{\zeta(s)} \tag{4}$$

Therefore the non-trivial zeros lie symmetrically to the real axis and the line $\Re(s) = \frac{1}{2}$.

In 1859 Riemann published the paper *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*. A translation of the paper is found in [2]. In the paper Riemann considers "very likely" that all the non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$. The statement

The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.

is known as the Riemann hypothesis.

2 Theorem

Lemma 2.1 *If*

$$\sum_{n=1}^{\infty} \frac{1}{n^{2\delta}} \frac{(-1)^{n-1}}{n^{1-z}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-z}} = 0 \tag{5}$$

where

$$\Re(z) = \frac{1}{2} + \delta \tag{6}$$

$$\Im(z) = t \neq 0 \tag{7}$$

and

$$-\frac{1}{2} < \delta < \frac{1}{2} \tag{8}$$

then

$$\delta = 0 \tag{9}$$

Proof:

Let be

$$\frac{(-1)^{n-1}}{n^{1-z}} = b_n + ic_n \qquad \text{for} \quad n = 1, 2, 3, \ldots \tag{10}$$

From (5) and (10) we obtain

$$\sum_{n=1}^{\infty} \frac{1}{n^{2\delta}} b_n = \sum_{n=1}^{\infty} b_n = 0 \tag{11}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^{2\delta}} c_n = \sum_{n=1}^{\infty} c_n = 0 \tag{12}$$

Since that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-z}} = \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)^{1-z}} - \frac{1}{(2n)^{1-z}} \right) \tag{13}$$

and

$$\frac{1}{(2n)^{1-z}} = \frac{(2n)^{it}}{(2n)^{\frac{1}{2}-\delta}} = \frac{1}{(2n)^{\frac{1}{2}-\delta}} \left[\cos(t \log(2n)) + i \sin(t \log(2n)) \right] \tag{14}$$

and

$$\frac{1}{(2n-1)^{1-z}} = \frac{(2n-1)^{it}}{(2n-1)^{\frac{1}{2}-\delta}} = \frac{1}{(2n-1)^{\frac{1}{2}-\delta}} \left[\cos(t \log(2n-1)) + i \sin(t \log(2n-1)) \right] \tag{15}$$

we obtain

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \left(\frac{\cos(t \log(2n-1))}{(2n-1)^{\frac{1}{2}-\delta}} - \frac{\cos(t \log(2n))}{(2n)^{\frac{1}{2}-\delta}} \right) \tag{16}$$

and

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \left(\frac{\sin(t \log(2n-1))}{(2n-1)^{\frac{1}{2}-\delta}} - \frac{\sin(t \log(2n))}{(2n)^{\frac{1}{2}-\delta}} \right) \tag{17}$$

Defining the functions

$$u(n) = \frac{\cos(t \log(2n-1))}{(2n-1)^{\frac{1}{2}}} \tag{18}$$

and

$$v(n) = \frac{\cos(t \log(2n))}{(2n)^{\frac{1}{2}}} \tag{19}$$

substituting (18) and (19) into (16) and using (11) we have

$$\sum_{n=1}^{\infty} \Big((2n-1)^{\delta} u(n) - (2n)^{\delta} v(n) \Big) = 0 \tag{20}$$

and

$$\sum_{n=1}^{\infty} n^{-2\delta} \Big((2n-1)^{\delta} u(n) - (2n)^{\delta} v(n) \Big) = 0 \tag{21}$$

The k-th part of (20) is

$$(2k-1)^{\delta}u(k)-(2k)^{\delta}v(k)= \\ -\sum_{n=1}^{k-1}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)-\sum_{n=k+1}^{\infty}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big) \tag{22}$$

where $k>1$. From (22) we can obtain the k-th part of (21)

$$k^{-2\delta}\Big((2k-1)^{\delta}u(k)-(2k)^{\delta}v(k)\Big)= \\ -\sum_{n=1}^{k-1}k^{-2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)-\sum_{n=k+1}^{\infty}k^{-2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big) \tag{23}$$

The left side of (23) obtained from (21) is

$$k^{-2\delta}\Big((2k-1)^{\delta}u(k)-(2k)^{\delta}v(k)\Big)= \\ -\sum_{n=1}^{k-1}n^{-2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)-\sum_{n=k+1}^{\infty}n^{-2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big) \tag{24}$$

Comparing (23) with (24) we conclude

$$\sum_{n=1}^{k-1}k^{-2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)+\sum_{n=k+1}^{\infty}k^{-2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)= \\ \sum_{n=1}^{k-1}n^{-2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)+\sum_{n=k+1}^{\infty}n^{-2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big) \tag{25}$$

for all $k>1$.

Let be $\delta>0$. Rearranging (25) we have

$$\sum_{n=1}^{k-1}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)+\sum_{n=k+1}^{\infty}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)= \\ \sum_{n=1}^{k-1}(\frac{k}{n})^{2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)+\sum_{n=k+1}^{\infty}(\frac{k}{n})^{2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big) \tag{26}$$

The limit of the left side of (26) when k tends to infinite is

$$\sum_{n=1}^{\infty}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big) \tag{27}$$

Let us notice that

$$(2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)=\frac{\cos(t\log(2n-1))}{(2n-1)^{\frac{1}{2}-\delta}}-\frac{\cos(t\log(2n))}{(2n)^{\frac{1}{2}-\delta}} \tag{28}$$

and

$$\lim_{n\to\infty}\left[\frac{\cos(t\log(2n-1))}{(2n-1)^{\frac{1}{2}-\delta}}-\frac{\cos(t\log(2n))}{(2n)^{\frac{1}{2}-\delta}}\right]=0 \tag{29}$$

for $0<\delta<\frac{1}{2}$. The limit of the right side of (26) when k tends to infinite is

$$\sum_{n=1}^{\infty}\lim_{k\to\infty}\left[\Big(\frac{k}{n}\Big)^{2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)\right] \tag{30}$$

Let us notice that

$$\lim_{n\to\infty}\left[\left(\frac{n-1}{n}\right)^{2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)\right]=0 \tag{31}$$

From (20), (27) and (30) we obtain

$$\sum_{n=1}^{\infty}\lim_{k\to\infty}\left[\left(\frac{k}{n}\right)^{2\delta}\Big((2n-1)^{\delta}u(n)-(2n)^{\delta}v(n)\Big)\right]=0 \tag{32}$$

It is clear that (32) is not correct because the series (30) does not converge for $0<\delta<\frac{1}{2}$. This means that δ is not great than 0.

Considering the symmetry conditions of the zeros of Riemann's zeta-function and that δ can not belongs to the interval $(0,\frac{1}{2})$, we conclude that δ can not belongs to $(-\frac{1}{2},0)$. Hence $\delta=0$.

$\square$

Theorem 2.1 *The non-trivial zeros of $\zeta(s)$ have real part equal to $\frac{1}{2}$.*

Proof:

Let us assume z to be such that $0<\Re(z)<1$, $\Im(z)\neq 0$ and

$$\zeta(z)=\zeta(1-z) \tag{33}$$

and

$$\zeta(z)=\zeta(\overline{z}) \tag{34}$$

If z is a non-trivial zero of ζ, then (33) and (34) are necessary conditions.

The Dirichlet series

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{s}}=(1-2^{1-s})\zeta(s) \tag{35}$$

is convergent for all values of s such that $\Re(s)>0$ [1]. For z we have

$$\zeta(z)=\frac{1}{(1-2^{1-z})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{z}} \tag{36}$$

From (33), (34) and (36) we obtain

$$\frac{1}{(1-2^{1-\overline{z}})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{z}}}=\frac{1}{(1-2^{z})}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-z}} \tag{37}$$

If z is a non-trivial zero of ζ, then

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\overline{z}}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{1-z}} \tag{38}$$

Let be

$$\Re(z)=\frac{1}{2}+\delta \qquad \text{and} \qquad \Im(z)=t \tag{39}$$

where

$$-\frac{1}{2}<\delta<\frac{1}{2} \tag{40}$$

Substituting (39) into (38) we have

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\frac{1}{2}+\delta-it}}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{\frac{1}{2}-\delta-it}} \tag{41}$$

or

$$\sum_{n=1}^{\infty} \frac{1}{n^{2\delta}} \frac{(-1)^{n-1}}{n^{1-z}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{1-z}} \tag{42}$$

Considering the lemma 2.1 we conclude

$$\Re(z) = \frac{1}{2} \tag{43}$$

□

References

[1] E. C. Titchmarsh and D. R. Heath-Brown, *The Theory of the Riemann Zeta-function*, Oxford University Press, (1988).

[2] H. M. Edwards, *Riemann's Zeta Function*, Academic Press, (1974).